11.7 Cylindrical and Spherical Coordinates

- Use cylindrical coordinates to represent surfaces in space.
- Use spherical coordinates to represent surfaces in space.

Cylindrical Coordinates

You have already seen that some two-dimensional graphs are easier to represent in polar coordinates than in rectangular coordinates. A similar situation exists for surfaces in space. In this section, you will study two alternative space-coordinate systems. The first, the **cylindrical coordinate system,** is an extension of polar coordinates in the plane to three-dimensional space.

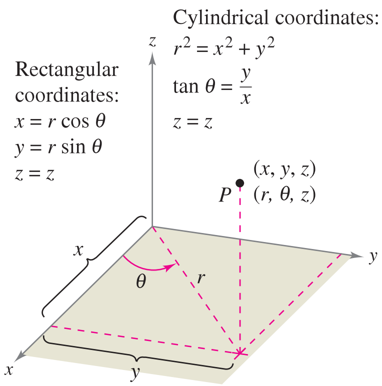


Figure 11.66

THE CYLINDRICAL COORDINATE SYSTEM

In a **cylindrical coordinate system,** a point P in space is represented by an ordered triple (r, θ, z).

1. (r, θ) is a polar representation of the projection of P in the xy-plane.
2. z is the directed distance from (r, θ) to P.

To convert from rectangular to cylindrical coordinates (or vice versa), use the following conversion guidelines for polar coordinates, as illustrated in Figure 11.66.

Cylindrical to rectangular:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

Rectangular to cylindrical:

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z$$

The point $(0, 0, 0)$ is called the **pole.** Moreover, because the representation of a point in the polar coordinate system is not unique, it follows that the representation in the cylindrical coordinate system is also not unique.

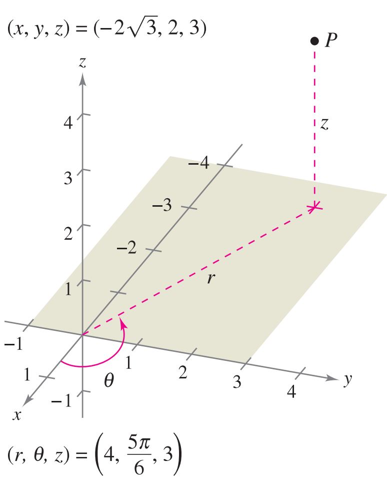


Figure 11.67

EXAMPLE 1 Converting from Cylindrical to Rectangular Coordinates

Convert the point $(r, \theta, z) = \left(4, \frac{5\pi}{6}, 3\right)$ to rectangular coordinates.

Solution Using the cylindrical-to-rectangular conversion equations produces

$$x = 4\cos\frac{5\pi}{6} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

$$y = 4\sin\frac{5\pi}{6} = 4\left(\frac{1}{2}\right) = 2$$

$$z = 3.$$

So, in rectangular coordinates, the point is $(x, y, z) = \left(-2\sqrt{3}, 2, 3\right)$, as shown in Figure 11.67. ■

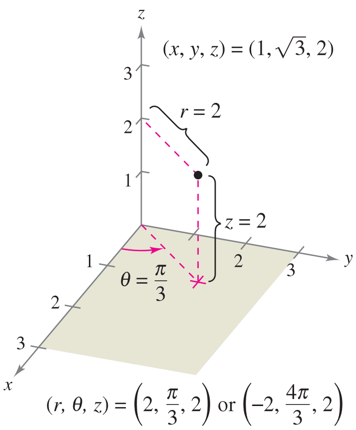


Figure 11.68

EXAMPLE 2 Converting from Rectangular to Cylindrical Coordinates

Convert the point $(x, y, z) = (1, \sqrt{3}, 2)$ to cylindrical coordinates.

Solution Use the rectangular-to-cylindrical conversion equations.

$$r = \pm\sqrt{1 + 3} = \pm 2$$

$$\tan\theta = \sqrt{3} \quad \Longrightarrow \quad \theta = \arctan(\sqrt{3}) + n\pi = \frac{\pi}{3} + n\pi$$

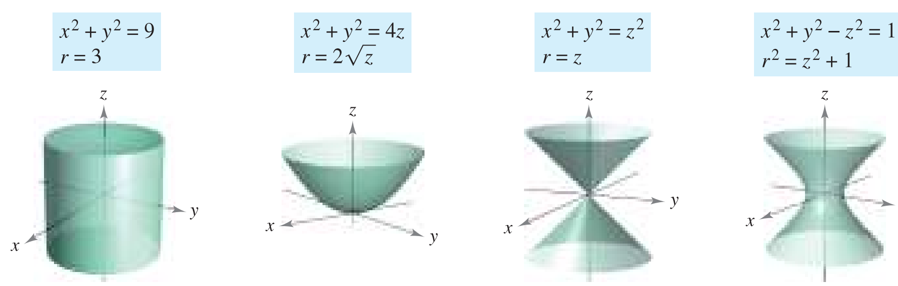
$$z = 2$$

You have two choices for r and infinitely many choices for θ. As shown in Figure 11.68, two convenient representations of the point are

$$\left(2, \frac{\pi}{3}, 2\right) \qquad r > 0 \text{ and } \theta \text{ in Quadrant I}$$

$$\left(-2, \frac{4\pi}{3}, 2\right). \qquad r < 0 \text{ and } \theta \text{ in Quadrant III}$$

Cylindrical coordinates are especially convenient for representing cylindrical surfaces and surfaces of revolution with the z-axis as the axis of symmetry, as shown in Figure 11.69.



Cylinder Paraboloid Cone Hyperboloid

Figure 11.69

Vertical planes containing the z-axis and horizontal planes also have simple cylindrical coordinate equations, as shown in Figure 11.70.

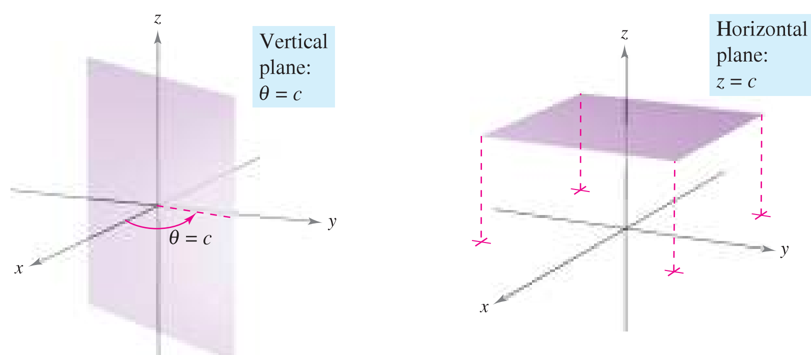


Figure 11.70

EXAMPLE 3 Rectangular-to-Cylindrical Conversion

Find an equation in cylindrical coordinates for the surface represented by each rectangular equation.

a. $x^2 + y^2 = 4z^2$

b. $y^2 = x$

Solution

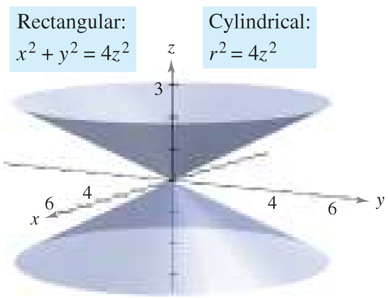


Figure 11.71

a. From the preceding section, you know that the graph $x^2 + y^2 = 4z^2$ is an elliptic cone with its axis along the z-axis, as shown in Figure 11.71. If you replace $x^2 + y^2$ with r^2, the equation in cylindrical coordinates is

$$x^2 + y^2 = 4z^2 \quad \text{Rectangular equation}$$
$$r^2 = 4z^2. \quad \text{Cylindrical equation}$$

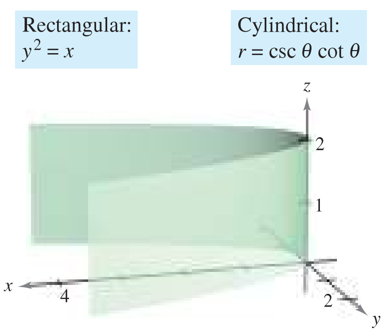


Figure 11.72

b. The graph of the surface $y^2 = x$ is a parabolic cylinder with rulings parallel to the z-axis, as shown in Figure 11.72. By replacing y^2 with $r^2 \sin^2\theta$ and x with $r\cos\theta$, you obtain the following equation in cylindrical coordinates.

$$y^2 = x \quad \text{Rectangular equation}$$
$$r^2 \sin^2\theta = r\cos\theta \quad \text{Substitute } r\sin\theta \text{ for } y \text{ and } r\cos\theta \text{ for } x.$$
$$r(r\sin^2\theta - \cos\theta) = 0 \quad \text{Collect terms and factor.}$$
$$r\sin^2\theta - \cos\theta = 0 \quad \text{Divide each side by } r.$$
$$r = \frac{\cos\theta}{\sin^2\theta} \quad \text{Solve for } r.$$
$$r = \csc\theta\cot\theta \quad \text{Cylindrical equation}$$

Note that this equation includes a point for which $r = 0$, so nothing was lost by dividing each side by the factor r. ■

Converting from cylindrical coordinates to rectangular coordinates is less straightforward than converting from rectangular coordinates to cylindrical coordinates, as demonstrated in Example 4.

EXAMPLE 4 Cylindrical-to-Rectangular Conversion

Find an equation in rectangular coordinates for the surface represented by the cylindrical equation

$$r^2 \cos 2\theta + z^2 + 1 = 0.$$

Solution

$$r^2\cos 2\theta + z^2 + 1 = 0 \quad \text{Cylindrical equation}$$
$$r^2(\cos^2\theta - \sin^2\theta) + z^2 + 1 = 0 \quad \text{Trigonometric identity}$$
$$r^2\cos^2\theta - r^2\sin^2\theta + z^2 = -1$$
$$x^2 - y^2 + z^2 = -1 \quad \text{Replace } r\cos\theta \text{ with } x \text{ and } r\sin\theta \text{ with } y.$$
$$y^2 - x^2 - z^2 = 1 \quad \text{Rectangular equation}$$

This is a hyperboloid of two sheets whose axis lies along the y-axis, as shown in Figure 11.73. ■

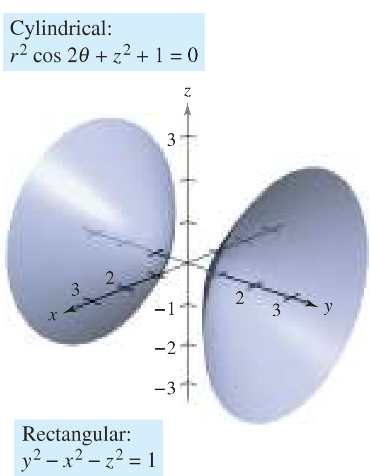


Figure 11.73

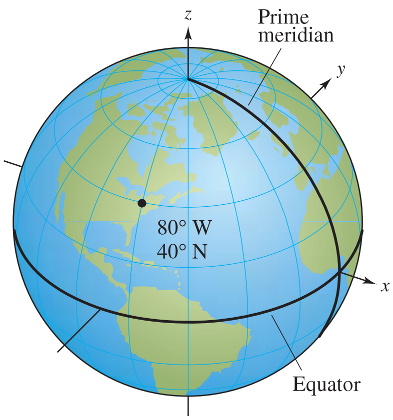


Figure 11.74

Spherical Coordinates

In the **spherical coordinate system,** each point is represented by an ordered triple: the first coordinate is a distance, and the second and third coordinates are angles. This system is similar to the latitude-longitude system used to identify points on the surface of Earth. For example, the point on the surface of Earth whose latitude is 40° North (of the equator) and whose longitude is 80° West (of the prime meridian) is shown in Figure 11.74. Assuming that the Earth is spherical and has a radius of 4000 miles, you would label this point as

$$(4000, -80°, 50°).$$

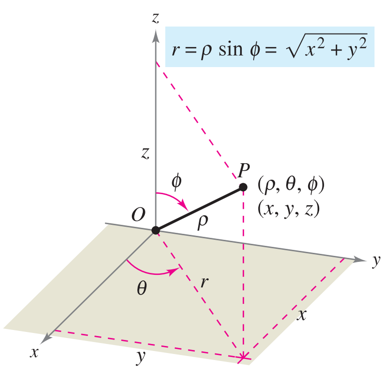
Radius — 80° clockwise from prime meridian — 50° down from North Pole

THE SPHERICAL COORDINATE SYSTEM

In a **spherical coordinate system,** a point P in space is represented by an ordered triple (ρ, θ, ϕ).

1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the same angle used in cylindrical coordinates for $r \geq 0$.
3. ϕ is the angle *between* the positive z-axis and the line segment $\overrightarrow{OP}$, $0 \leq \phi \leq \pi$.

Note that the first and third coordinates, ρ and ϕ, are nonnegative. ρ is the lowercase Greek letter *rho*, and ϕ is the lowercase Greek letter *phi*.



Spherical coordinates
Figure 11.75

The relationship between rectangular and spherical coordinates is illustrated in Figure 11.75. To convert from one system to the other, use the following.

Spherical to rectangular:

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi$$

Rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan \theta = \frac{y}{x}, \qquad \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

To change coordinates between the cylindrical and spherical systems, use the following.

Spherical to cylindrical* (r ≥ 0)*:

$$r^2 = \rho^2 \sin^2 \phi, \qquad \theta = \theta, \qquad z = \rho \cos \phi$$

Cylindrical to spherical* (r ≥ 0)*:

$$\rho = \sqrt{r^2 + z^2}, \qquad \theta = \theta, \qquad \phi = \arccos\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$$

The spherical coordinate system is useful primarily for surfaces in space that have a *point* or *center* of symmetry. For example, Figure 11.76 shows three surfaces with simple spherical equations.

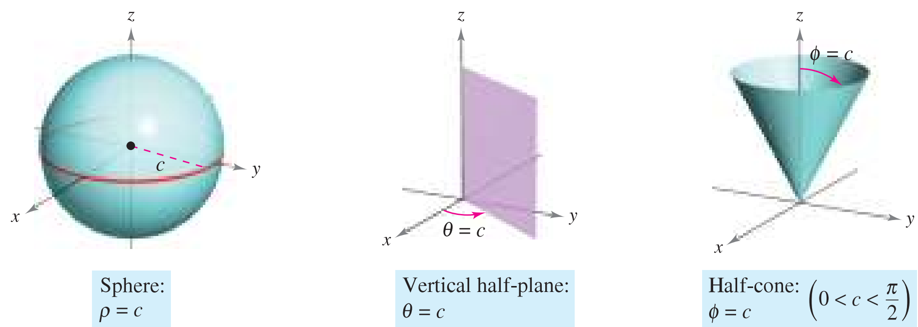


Figure 11.76

EXAMPLE 5 Rectangular-to-Spherical Conversion

Find an equation in spherical coordinates for the surface represented by each rectangular equation.

a. Cone: $x^2 + y^2 = z^2$

b. Sphere: $x^2 + y^2 + z^2 - 4z = 0$

Solution

a. Making the appropriate replacements for x, y, and z in the given equation yields the following.

$$\begin{aligned} x^2 + y^2 &= z^2 \\ \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta &= \rho^2 \cos^2 \phi \\ \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) &= \rho^2 \cos^2 \phi \\ \rho^2 \sin^2 \phi &= \rho^2 \cos^2 \phi \\ \frac{\sin^2 \phi}{\cos^2 \phi} &= 1 \qquad \rho \geq 0 \\ \tan^2 \phi &= 1 \qquad \phi = \pi/4 \text{ or } \phi = 3\pi/4 \end{aligned}$$

The equation $\phi = \pi/4$ represents the *upper* half-cone, and the equation $\phi = 3\pi/4$ represents the *lower* half-cone.

b. Because $\rho^2 = x^2 + y^2 + z^2$ and $z = \rho \cos \phi$, the given equation has the following spherical form.

$$\rho^2 - 4\rho \cos \phi = 0 \quad \Rightarrow \quad \rho(\rho - 4 \cos \phi) = 0$$

Temporarily discarding the possibility that $\rho = 0$, you have the spherical equation

$$\rho - 4 \cos \phi = 0 \quad \text{or} \quad \rho = 4 \cos \phi.$$

Note that the solution set for this equation includes a point for which $\rho = 0$, so nothing is lost by discarding the factor ρ. The sphere represented by the equation $\rho = 4 \cos \phi$ is shown in Figure 11.77. ■

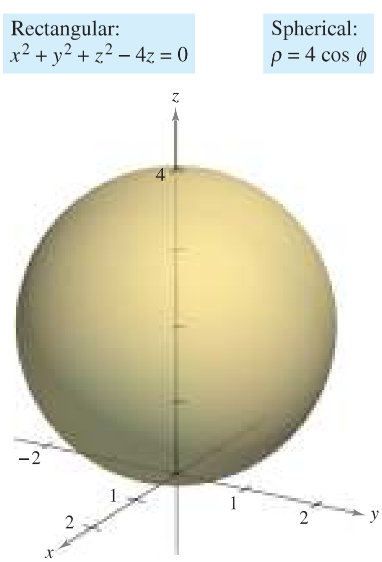


Figure 11.77

11.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, convert the point from cylindrical coordinates to rectangular coordinates.

1. $(-7, 0, 5)$
2. $(2, -\pi, -4)$
3. $(3, \pi/4, 1)$
4. $(6, -\pi/4, 2)$
5. $(4, 7\pi/6, 3)$
6. $(-0.5, 4\pi/3, 8)$

In Exercises 7–12, convert the point from rectangular coordinates to cylindrical coordinates.

7. $(0, 5, 1)$
8. $(2\sqrt{2}, -2\sqrt{2}, 4)$
9. $(2, -2, -4)$
10. $(3, -3, 7)$
11. $(1, \sqrt{3}, 4)$
12. $(2\sqrt{3}, -2, 6)$

In Exercises 13–20, find an equation in cylindrical coordinates for the equation given in rectangular coordinates.

13. $z = 4$
14. $x = 9$
15. $x^2 + y^2 + z^2 = 17$
16. $z = x^2 + y^2 - 11$
17. $y = x^2$
18. $x^2 + y^2 = 8x$
19. $y^2 = 10 - z^2$
20. $x^2 + y^2 + z^2 - 3z = 0$

In Exercises 21–28, find an equation in rectangular coordinates for the equation given in cylindrical coordinates, and sketch its graph.

21. $r = 3$
22. $z = 2$
23. $\theta = \pi/6$
24. $r = \frac{1}{2}z$
25. $r^2 + z^2 = 5$
26. $z = r^2 \cos^2 \theta$
27. $r = 2 \sin \theta$
28. $r = 2 \cos \theta$

In Exercises 29–34, convert the point from rectangular coordinates to spherical coordinates.

29. $(4, 0, 0)$
30. $(-4, 0, 0)$
31. $(-2, 2\sqrt{3}, 4)$
32. $(2, 2, 4\sqrt{2})$
33. $(\sqrt{3}, 1, 2\sqrt{3})$
34. $(-1, 2, 1)$

In Exercises 35–40, convert the point from spherical coordinates to rectangular coordinates.

35. $(4, \pi/6, \pi/4)$
36. $(12, 3\pi/4, \pi/9)$
37. $(12, -\pi/4, 0)$
38. $(9, \pi/4, \pi)$
39. $(5, \pi/4, 3\pi/4)$
40. $(6, \pi, \pi/2)$

In Exercises 41–48, find an equation in spherical coordinates for the equation given in rectangular coordinates.

41. $y = 2$
42. $z = 6$
43. $x^2 + y^2 + z^2 = 49$
44. $x^2 + y^2 - 3z^2 = 0$
45. $x^2 + y^2 = 16$
46. $x = 13$
47. $x^2 + y^2 = 2z^2$
48. $x^2 + y^2 + z^2 - 9z = 0$

In Exercises 49–56, find an equation in rectangular coordinates for the equation given in spherical coordinates, and sketch its graph.

49. $\rho = 5$
50. $\theta = \dfrac{3\pi}{4}$
51. $\phi = \dfrac{\pi}{6}$
52. $\phi = \dfrac{\pi}{2}$
53. $\rho = 4 \cos \phi$
54. $\rho = 2 \sec \phi$
55. $\rho = \csc \phi$
56. $\rho = 4 \csc \phi \sec \theta$

In Exercises 57–64, convert the point from cylindrical coordinates to spherical coordinates.

57. $(4, \pi/4, 0)$
58. $(3, -\pi/4, 0)$
59. $(4, \pi/2, 4)$
60. $(2, 2\pi/3, -2)$
61. $(4, -\pi/6, 6)$
62. $(-4, \pi/3, 4)$
63. $(12, \pi, 5)$
64. $(4, \pi/2, 3)$

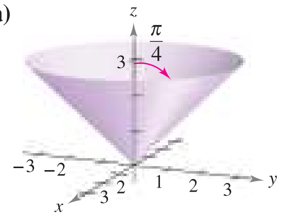
In Exercises 65–72, convert the point from spherical coordinates to cylindrical coordinates.

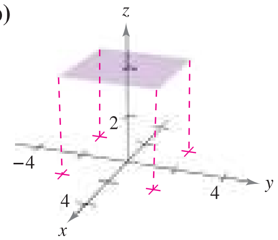
65. $(10, \pi/6, \pi/2)$
66. $(4, \pi/18, \pi/2)$
67. $(36, \pi, \pi/2)$
68. $(18, \pi/3, \pi/3)$
69. $(6, -\pi/6, \pi/3)$
70. $(5, -5\pi/6, \pi)$
71. $(8, 7\pi/6, \pi/6)$
72. $(7, \pi/4, 3\pi/4)$

CAS **In Exercises 73–88, use a computer algebra system or graphing utility to convert the point from one system to another among the rectangular, cylindrical, and spherical coordinate systems.**

	Rectangular	*Cylindrical*	*Spherical*
73.	$(4, 6, 3)$		
74.	$(6, -2, -3)$		
75.		$(5, \pi/9, 8)$	
76.		$(10, -0.75, 6)$	
77.			$(20, 2\pi/3, \pi/4)$
78.			$(7.5, 0.25, 1)$
79.	$(3, -2, 2)$		
80.	$(3\sqrt{2}, 3\sqrt{2}, -3)$		
81.	$(5/2, 4/3, -3/2)$		
82.	$(0, -5, 4)$		
83.		$(5, 3\pi/4, -5)$	
84.		$(-2, 11\pi/6, 3)$	
85.		$(-3.5, 2.5, 6)$	
86.		$(8.25, 1.3, -4)$	
87.			$(3, 3\pi/4, \pi/3)$
88.			$(8, -\pi/6, \pi)$

In Exercises 89–94, match the equation (written in terms of cylindrical or spherical coordinates) with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)


(b)


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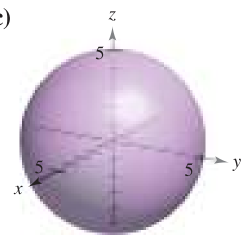

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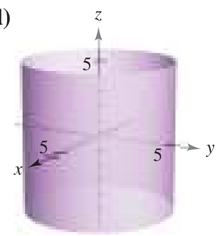

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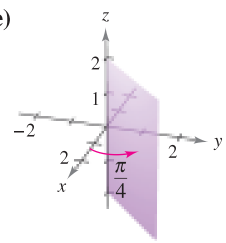
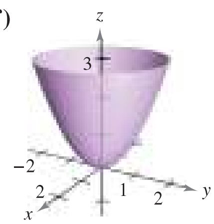


(f)


89. $r = 5$

90. $\theta = \frac{\pi}{4}$

91. $\rho = 5$

92. $\phi = \frac{\pi}{4}$

93. $r^2 = z$

94. $\rho = 4 \sec \phi$

WRITING ABOUT CONCEPTS

95. Give the equations for the coordinate conversion from rectangular to cylindrical coordinates and vice versa.

96. Explain why in spherical coordinates the graph of $\theta = c$ is a half-plane and not an entire plane.

97. Give the equations for the coordinate conversion from rectangular to spherical coordinates and vice versa.

CAPSTONE

98. (a) For constants a, b, and c, describe the graphs of the equations $r = a$, $\theta = b$, and $z = c$ in cylindrical coordinates.

(b) For constants a, b, and c, describe the graphs of the equations $\rho = a$, $\theta = b$, and $\phi = c$ in spherical coordinates.

In Exercises 99–106, convert the rectangular equation to an equation in (a) cylindrical coordinates and (b) spherical coordinates.

99. $x^2 + y^2 + z^2 = 25$

100. $4(x^2 + y^2) = z^2$

101. $x^2 + y^2 + z^2 - 2z = 0$

102. $x^2 + y^2 = z$

103. $x^2 + y^2 = 4y$

104. $x^2 + y^2 = 36$

105. $x^2 - y^2 = 9$

106. $y = 4$

In Exercises 107–110, sketch the solid that has the given description in cylindrical coordinates.

107. $0 \le \theta \le \pi/2,\ 0 \le r \le 2,\ 0 \le z \le 4$

108. $-\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 3,\ 0 \le z \le r\cos\theta$

109. $0 \le \theta \le 2\pi,\ 0 \le r \le a,\ r \le z \le a$

110. $0 \le \theta \le 2\pi,\ 2 \le r \le 4,\ z^2 \le -r^2 + 6r - 8$

In Exercises 111–114, sketch the solid that has the given description in spherical coordinates.

111. $0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/6,\ 0 \le \rho \le a \sec \phi$

112. $0 \le \theta \le 2\pi,\ \pi/4 \le \phi \le \pi/2,\ 0 \le \rho \le 1$

113. $0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/2,\ 0 \le \rho \le 2$

114. $0 \le \theta \le \pi,\ 0 \le \phi \le \pi/2,\ 1 \le \rho \le 3$

Think About It **In Exercises 115–120, find inequalities that describe the solid, and state the coordinate system used. Position the solid on the coordinate system such that the inequalities are as simple as possible.**

115. A cube with each edge 10 centimeters long

116. A cylindrical shell 8 meters long with an inside diameter of 0.75 meter and an outside diameter of 1.25 meters

117. A spherical shell with inside and outside radii of 4 inches and 6 inches, respectively

118. The solid that remains after a hole 1 inch in diameter is drilled through the center of a sphere 6 inches in diameter

119. The solid inside both $x^2 + y^2 + z^2 = 9$ and $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

120. The solid between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$, and inside the cone $z^2 = x^2 + y^2$

True or False? **In Exercises 121–124, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

121. In cylindrical coordinates, the equation $r = z$ is a cylinder.

122. The equations $\rho = 2$ and $x^2 + y^2 + z^2 = 4$ represent the same surface.

123. The cylindrical coordinates of a point (x, y, z) are unique.

124. The spherical coordinates of a point (x, y, z) are unique.

125. Identify the curve of intersection of the surfaces (in cylindrical coordinates) $z = \sin\theta$ and $r = 1$.

126. Identify the curve of intersection of the surfaces (in spherical coordinates) $\rho = 2 \sec \phi$ and $\rho = 4$.